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Diachronic Quantum Action Principle

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Classical path and action are diachronic concepts in that they refer to many times instead of just one. The concept of a path is quantized into the concept of a propagation process between the initial preparation of and a measurement on a quantum system. A new quantum action is defined as a linear operator on the space of propagation processes, analogously to representing observables as linear operators on the ket and bra spaces. This quantization of paths and action results in a diachronic action principle: a variation of a dynamical propagation process is generated by the associated variation of the quantum action. The form of this principle is a candidate for the form of a dynamical principle of a theory without a classical time parameter.

1. INTRODUCTION

A central problem today is the synthesis of general relativity and quantum theory. In general relativity we require a theory to be independent of coordinate systems. As a first step to formulate a quantum theory that is independent of coordinates, we seek a quantum theory that is independent of pictures. Moreover, we expect a quantum theory of space-time and matter that employs no classical coordinates. One of the first questions then is, What would dynamics mean in the absence of classical time coordinates (Finkelstein, 1972a, Section II)? We give a candidate for the form of a dynamical law of such a theory.

We refer to the object of an experiment as "the quantum system." In Sections 2 and 3 we analyze quantum mechanical experiments into modes of becoming, namely direct acts by the experimenter on the quantum system and propagation of it. Possible propagations are represented by elements of the "process space." The sequence of direct acts of the experimenter during

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an experiment is represented by an element of the dual "coprocess space." In order to represent "processes" (e.g., propagations) and "coprocesses" (e.g., sequences of experimental acts) we use discretizations of the time axis, similar to Feynman's (1948) space-time approach to quantum mechanics. The process and coprocess spaces then are bundles whose fibers are tensor products of ket and bra spaces at various times. We assume no particular picture in that we do not identify ket (or bra) spaces at different times.

The propagation tensor is a tensor product of propagations over time steps which represents the *process* of propagation rather completely. The contraction of this tensor is the picture-independent version of the usual evolution operator, which represents the net effect of a propagation (that is not interrupted by selective filter acts of the experimenter) on a transition amplitude. Feynman's integral and Schwinger's differential action principles refer to matrix elements of the evolution operator. In Section 4 we use Feynman's principle to derive a differential action principle for the quantum mechanical propagation tensor. The action in this diachronic principle is a linear operator on the process space. In Section 5 we derive the picture-independent form of Schwinger's action principle from our diachronic one, in order to illustrate the relationship between the two.

Section 6 compares the three action principles and recalls the meaning that the present paper assigns to action in quantum theories, including theories without a classical time parameter.

A point $\mathfrak{t}$ of the time axis $\mathbb{T}$ often will be denoted by its coordinate representation with respect to a particular time variable t:

$$\mathfrak{t} \leftrightarrow t{:}t' \tag{1.1}$$

The statement that the variable t takes on the value t' is abbreviated as "$t{:}t'$."

2. PICTURE-INDEPENDENT LANGUAGE OF QUANTUM MECHANICS

An experimenter starts and ends an experiment on a quantum system by certain operations he or she performs on that system. These two operations have, respectively, been called "preparation of the system in a state" and "test" by Giles (1970, Section 3),[2] "preparation" and "registration" by Ludwig

[2]Giles points out that in an elementary experiment one first prepares a system in a state x and second applies a test a to this system. He represents the state x by a density operator D_x and the test a by a projector A_a. For that representation one needs to employ the adjoint operator $\dagger$. Giles does not consider time evolution. The experiments we consider involve time evolution and are assumed to be ideally sharp in that the preparations (input operations) and the tests (outtake operations) are maximally precise. Due to this precision the first and second stages can be represented by a ket and a bra, respectively, which does not require the $\dagger$. The input and outtake operations and possible other acts of the experimenter between them we represent jointly by a coprocess tensor. We could represent imprecise acts of the experiment by associating probabilities with different coprocess tensors.

(1983, 1985), and "initial" and "final actions" or "input" and "outtake operations" by Finkelstein (1995, §1.2.1). We follow Finkelstein in that sharp (maximally specified) input and outtake operations are represented by elements of dual vector spaces. These elements are called ket and bra vectors, respectively. In contrast, most textbooks today say that kets and bras indiscriminately represent system "states," in the sense of states of being rather than modes of preparation.

The simplest kind of experiment consists of an input and an outtake operation performed in immediate succession. This can be illustrated very well by the example of polarization experiments. It is well known that a Malus polarization experiment consists of two stages: *two* polarizers are held in a stream of photons, not just one. The input operation consists of holding the first polarizer in the photon stream. The input of the experiment consists of the photons that have passed through this polarizer. The outtake operation consists of holding the second polarizer in this stream of input photons. A photon detector is mounted behind the second polarizer. The outtake of the experiment includes the photons that pass the second polarizer, too. A transition from the input to the outtake phase here means the passing of an input photon through the second polarizer. The transition probability is the ratio of the number of outtake to the number of input systems, here photons.

The input operation at some time t is represented and denoted by a ket $|\psi\rangle$ in a complex vector space $I(\mathrm{t})$, the outtake operation by a bra $\langle\varphi|$ in the dual space $I(\mathrm{t})^{\mathrm{D}}$. The many-time input and outtake spaces,

$$I\mathbb{T} = \bigcup_{\mathrm{t}\in\mathbb{T}} I(\mathrm{t}) \cong I \times \mathbb{T}$$

$$I^{\mathrm{D}}\mathbb{T} = \bigcup_{\mathrm{t}\in\mathbb{T}} I(\mathrm{t})^{\mathrm{D}} \cong I^{\mathrm{D}} \times \mathbb{T} \tag{2.1}$$

are trivial fiber bundles (e.g., Nash and Sen, 1983) over the base space $\mathbb{T}$ with a common structure group G. The fibers $I(\mathrm{t}_r)$ and $I^{\mathrm{D}}(\mathrm{t}_r)$ over an arbitrarily chosen reference time t_r serve as standard fibers. The triviality of the bundles would allow one to define each bundle by a single global trivialization [factorization like (2.1)], in which case G would be trivial. However, this would single out a particular isomorphism of, say, input spaces at different times. Such an identification is avoided here, because nonsimultaneous input operations are regarded as different, as one does with nonsimultaneous events in relativity. The dual statements hold for the outtake operations.

For the definition of $I\mathbb{T}$ note that we restrict ourselves to global trivializations. In quantum mechanics such a trivialization is called a picture. A connection π transports a ket $|\psi\rangle \in I(\mathrm{t}_1)$ at time t_1 into kets $\pi(\mathrm{t}_2, \mathrm{t}_1)|\psi\rangle \in$

$I(t_2)$ at other times t_2, dually a bra $\langle\varphi| \in I(t_1)$ into $\langle\varphi|\pi(t_1, t_2) \in I^D(t_2)$, and has the composition property

$$\pi(t_3, t_1) = \pi(t_3, t_2)\pi(t_2, t_1) \tag{2.2}$$

A connection π induces the following picture isomorphisms on $I\mathbb{T}$ and $I^D\mathbb{T}$:

$$\begin{aligned} B(\pi)\colon \quad & I\mathbb{T} \to I(t_r) \times \mathbb{T} \\ & |\psi\rangle \in I(t) \mapsto (\pi(t_r, t)|\psi\rangle, t) =: (|\psi\rangle^\pi, t) =: |\psi, t\rangle^\pi \\ B(\pi)\colon \quad & I^D\mathbb{T} \to I^D(t_r) \times \mathbb{T} \\ & \langle\varphi| \in I(t) \mapsto (\langle\varphi|\pi(t, t_r), t) =: ({}^\pi\langle\varphi|, t) =: {}^\pi\langle\varphi, t| \end{aligned} \tag{2.3}$$

where t_r is the arbitrarily chosen reference time. ($|\psi, t\rangle$ and $\langle\varphi, t|$ are not functions of time. t just denotes at which time the input and outtake operations are to be performed.) Let π and $\bar{\pi}$ be arbitrary connections. For fixed t, $\bar{\pi}(t_r, t) \circ \pi(t, t_r)$ is a transition function (e.g., Nash and Sen, 1983) of the bundle $I\mathbb{T}$. The structure group G of $I\mathbb{T}$ consists of the transition functions at the various times between the various pictures $B(\pi)$ and $B(\bar{\pi})$,

$$G = \{\bar{\pi}(t_r, t)\pi(t, t_r)\} \tag{2.4}$$

Dually, $\pi(t_r, t) \circ \bar{\pi}(t, t_r)$ are the transition functions of the bundle $I^D\mathbb{T}$ and form a structure group isomorphic to G. Below we shall use trivial fiber bundles over (a subset of a power of) $\mathbb{T}$, with standard fibers built by tensor products of I and I^D and with the same structure group G. A ket (bra) connection induces a picture of such a bundle similarly as the picture of the ket and bra bundles were constructed in equation (2.3).

A many-time linear operator

$$O = \{O(t), t \in \mathbb{T}\} \tag{2.5}$$

is a section of the fiber bundle

$$\mathcal{L}\mathbb{T} = \bigcup_{t\in\mathbb{T}} I(t) \otimes I^D(t) \tag{2.6}$$

A classical time variable t is a map

$$\begin{aligned} t\colon \quad & \mathbb{T} \to \mathbb{R} \\ & t \mapsto t' = t(t) \end{aligned} \tag{2.7}$$

It induces the map

$$\begin{aligned} t\colon \quad & I\mathbb{T} \to I\mathbb{T} \\ & |\psi\rangle \in I(t) \mapsto t|\psi\rangle := t(t)|\psi\rangle \in I(t) \end{aligned} \tag{2.8}$$

and dually for the bra vectors.

The derivative of a many-time operator O with respect to a time variable t and a connection is the covariant derivative

$$\dot{O}_{t,\pi}(t{:}t') \equiv \left(\frac{\nabla}{dt}\right)_{\pi} O(t{:}t') := \left.\frac{d}{dt''}\right|_{t''=t'} \pi(t{:}t',\, t{:}t'')O(t{:}t'')\pi(t{:}t'',\, t{:}t') \quad (2.9)$$

In the picture $B(\pi)$ a many-time operator O and a connection $\bar{\pi}$ are represented by

$$\begin{aligned} O^{\pi}(\mathrm{t}) &= \pi(\mathrm{t}_r,\, \mathrm{t})O(\mathrm{t})\pi(\mathrm{t},\, \mathrm{t}_r), \qquad \bar{\pi}^{\pi}(\mathrm{t}_2,\, \mathrm{t}_1) = \pi(\mathrm{t}_r,\, \mathrm{t}_2)\bar{\pi}(\mathrm{t}_2,\, \mathrm{t}_1)\pi(\mathrm{t}_1,\, \mathrm{t}_r) \\ &\in \mathscr{L}(\mathrm{t}_r) = I(\mathrm{t}_r) \otimes I^{\mathrm{D}}(\mathrm{t}_r) \end{aligned} \quad (2.10)$$

For small time steps from t_1 to t_2 we can write

$$\bar{\pi}^{\pi}(t{:}t' + \Delta t',\, t{:}t') = 1 - \frac{i}{\hbar}\,\Gamma(\bar{\pi})^{t,\pi}(t')\Delta t' \quad (2.11)$$

$\Gamma(\bar{\pi})^{t,\pi}$ is the connection coefficient of $\bar{\pi}$ with respect to the time variable t and with respect to the picture $B(\pi)$. The coefficient of the connection π vanishes in the picture $B(\pi)$,

$$\pi^{\pi} = 1, \qquad \Gamma(\pi)^{t,\pi} = 0 \quad (2.12)$$

The picture representation of equation (2.9) is

$$\left[\left(\frac{\nabla}{d\bar{t}}\right)_{\bar{\pi}} O(t{:}t')\right]^{t,\pi} = \frac{dt}{d\bar{t}}\left(\frac{dO(t{:}t')}{dt'} + \frac{i}{\hbar}\,[\Gamma(\bar{\pi})^{t,\pi}(t'),\, O(t{:}t')^{\pi}]\right) \quad (2.13)$$

The propagation process between the input and outtake operations is represented by a connection U (Asorey *et al.*, 1982). U, the associated covariant derivative, and the picture $B(U)$ (which is the usual Heisenberg picture) are called dynamical. We assume a positive-definite metric of the ket spaces. The associated adjoint operator † maps kets into bras, and vice versa, and is antiunitary,

$$\langle\psi| \in I^{\mathrm{D}}(\mathrm{t}) \overset{\dagger}{\leftrightarrow} |\psi\rangle \in I(\mathrm{t}) \quad (2.14)$$

$$\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^{*} \overset{\dagger}{\leftrightarrow} \langle\varphi|\psi\rangle \quad (2.15)$$

$|\psi\rangle$ and $\langle\psi|$, as well as the operations they represent, we call adjoint to one another. The transition of a quantum system from an input phase of an experiment to the adjoint outtake phase is compulsory. Higher order tensors transform according to the scheme

$$I(\mathrm{t}_2) \otimes I^{\mathrm{D}}(\mathrm{t}_1) \overset{\dagger}{\leftrightarrow} I(\mathrm{t}_1) \otimes I^{\mathrm{D}}(\mathrm{t}_2)$$

$$O^\dagger = (O^n_m)^* |a^m\rangle\langle b_n| \overset{\dagger}{\leftrightarrow} O = O^n_m |b_n\rangle\langle a^m| \tag{2.16}$$

The dynamical connection is assumed to be unitary

$$U(\mathrm{t}_2, \mathrm{t}_1)^\dagger U(\mathrm{t}_2, \mathrm{t}_1) = 1 \tag{2.17}$$

We restrict ourselves to unitary pictures $B(\pi)$, which correspond to unitary connections π. A ket and its adjoint bra at some time then are represented by a ket and its adjoint bra at the reference time t_r. This means that the many-time adjoint operation is represented by the restriction of $\dagger$ on $I(\mathrm{t}_r)$ and $I(\mathrm{t}_r)^{\mathrm{D}}$,

$$\dagger^\pi(\mathrm{t}) = \dagger(\mathrm{t}_r) \tag{2.18}$$

The structure group, equation (2.4), then is the group of unitary operators on $I(\mathrm{t}_r)$,

$$G = \{u \in I(\mathrm{t}_r) \otimes I^{\mathrm{D}}(\mathrm{t}_r) | u^{\dagger(\mathrm{t}_r)} u = 1\} =: \mathcal{U} \tag{2.19}$$

3. DIACHRONIC LANGUAGE OF QUANTUM MECHANICS

All the considered experiments shall occur during a time interval $[\mathrm{t}_i, \mathrm{t}_f]$; we restrict the time axis accordingly,

$$\mathbb{T} = [\mathrm{t}_i, \mathrm{t}_f] \subset \mathbb{R} \tag{3.1}$$

With respect to a useful time variable t this interval has the length

$$T' = t(\mathrm{t}_f) - t(\mathrm{t}_i) \tag{3.2}$$

We start an experiment on a quantum system by preparing the system at some time t_r (input operation) and end it by testing for a particular system property (output operation) at some time t_N. At intermediate times t_n we may act with filters on the quantum system, which are represented by projectors P ($P^\dagger = P = P^2$), or we may shift the quantum system with respect to a particular quantum variable (e.g., by adding an additional particle (or excitation) to the experimental region or by absorbing one). Both kinds of acts are represented by elements of $I(\mathrm{t}_n) \otimes I^{\mathrm{D}}(\mathrm{t}_n)$. The passive act is represented by the unity operator. We restrict ourselves here to experiments that can be described in terms of a fixed number $N - 1$ of intermediate times t_n, with

$$\Delta t_n' = t(\mathrm{t}_n) - t(\mathrm{t}_{n-1}) < 2\,\frac{T'}{N} \tag{3.3}$$

(The number 2 on the right side could be replaced by any number independent of N and larger than 1. Here n is the index of the time step from t_{n-1} to t_n.) We choose the number N of time steps so large that our restriction is not severe. The sequence of direct acts of the experimenter, viz. input operation, intermediate acts, and outtake operation, is an example of a coprocess and is represented by an element σ in the coprocess space,

$$\sigma = \langle\varphi(\mathrm{t}_N)\big| \otimes O_{N-1}(\mathrm{t}_{N-1}) \otimes \cdots \otimes O_1(\mathrm{t}_1) \otimes \big|\psi(\mathrm{t}_0)\rangle \tag{3.4}$$

$$\in \Pi^{\mathrm{D}}\mathbb{T}^{N+1} := \bigcup_{\mathrm{z}_n \in \mathbb{T} \mid t(\mathrm{z}_n) - t(\mathrm{z}_{n-1}) < 2\,T'/N} I^{\mathrm{D}}(\mathrm{z}_N)$$

$$\otimes \cdots \otimes I(\mathrm{z}_1) \otimes I^{\mathrm{D}}(\mathrm{z}_1) \otimes I(\mathrm{z}_0) \tag{3.5}$$

(t_0, t_1, etc., do not denote a functional dependence of $|\psi(\mathrm{t}_0)\rangle$, $O_1(\mathrm{t}_1)$, etc., but just the times to which the ket, the linear operator, etc., are attached.)

For a particular sequence of act times t_n we represent a possible propagation process by a tensor product

$$\pi = \pi(\mathrm{t}_N, \mathrm{t}_{N-1}) \otimes \cdots \otimes \pi(\mathrm{t}_1, \mathrm{t}_0) \tag{3.6}$$

$$\in \Pi\mathbb{T}^{N+1} := \bigcup_{\mathrm{z}_n \in \mathbb{T} \mid t(\mathrm{z}_n) - t(\mathrm{z}_{n-1}) < 2\,T'/N} I(\mathrm{z}_N)$$

$$\otimes \cdots \otimes I^{\mathrm{D}}(\mathrm{z}_1) \otimes I(\mathrm{z}_1) \otimes I^{\mathrm{D}}(\mathrm{z}_0) \tag{3.7}$$

[Compare with Finkelstein's path tensor (1995, §12.5.2) or process stator (1972b, p. 2931).]

A "quantum frame" α is a system of orthogonal bases $\{|a', \mathrm{t}\rangle\}$,

$$\langle a'', \mathrm{t}\big|a', \mathrm{t}\rangle = \delta_{a'',a'} \tag{3.8a}$$

of the ket fibers at different times, the dual bra bases, and the induced product bases of the fibers of the process and coprocess spaces. The induced bases have the elements

$$\alpha(a'(N), \mathrm{t}_N, \ldots, a(2), a'(1), \mathrm{t}_1, a(1), \mathrm{t}_0)$$

$$:= \otimes \prod_{n=1}^{N} \big|a'(n), \mathrm{t}_n\rangle\langle a(n), \mathrm{t}_{n-1}\big| \in \Pi(\mathrm{t}_N, \ldots, \mathrm{t}_0) \tag{3.8b}$$

$$\alpha^{\mathrm{D}}(a'(N), \mathrm{t}_N, \ldots, a(2), a'(1), \mathrm{t}_1, a(1), \mathrm{t}_0)$$

$$:= \prod_{n=1}^{N} \langle a'(n), \mathrm{t}_n\big| \otimes \big|a(n), \mathrm{t}_{n-1}\rangle \in \Pi^{D}(\mathrm{t}_N, \ldots, \mathrm{t}_0) \tag{3.8c}$$

Typically one uses basis vectors that are eigenvectors of a complete sequence of observables. One can interpret a basis process, equation (3.8b), as a succession of particular ways to annihilate and recreate the quantum system, a general process (represented in the process space $\Pi\mathbb{T}^{N+1}$) as a superposition of such successions. Mathematically this superposition property is represented by the linearity of the process space. There is becoming but no being; a quantum *is* not, it decays at one instant and (in our time-step approximation of propagation) reemerges at the next considered instant. The explicit representation (3.6) of the *process* of propagation may prepare us for a joint quantization program of space-time and dynamics, such as suggested by Finkelstein (1972a,b, 1995). According to the cited program the concept of a system evolving in time is expressed by a quantum composition of quantum processes, such as elementary processes called chronons.

A tensor product of a coprocess and a process tensor, such as

$$\sigma \otimes \pi = \langle \varphi(t_N) | \otimes \pi(t_N, t_{N-1}) \otimes O_{N-1}(t_{N-1}) \\ \otimes \cdots \otimes O_1(t_1) \otimes \pi(t_1, t_0) \otimes | \psi(t_0) \rangle \tag{3.9}$$

represents an experiment completely. We call this description diachronic in the sense that it extends over many times. The transition amplitude (probability amplitude for the quantum system to pass from the input phase, via the propagation steps, through the medial acts into the outtake phase) is the tensor contraction

$$\sigma\pi = \mathrm{Tr}(\sigma \otimes \pi) \tag{3.10}$$

If the transition occurs, we say that the experiment had a positive result or that the process combination $\sigma \otimes \pi$ occurred.

The activity of the experimenter affects the endosystem even during the propagation steps: she or he may control the experimental background such as the properties of an optically active medium in a polarization experiment or the value of an external field or source. Because there always is an experimental background, the separation of an experiment into a coprocess (direct acts of the experimenter) and a propagation process is not unique: a particle the experimenter adds or absorbs at an intermediate time t_n could equivalently have been emitted or absorbed by an external source during the preceding propagation step.

The concepts of input and outtake operations are related to the concepts of pre- and postselection (past and future measurements) developed by Aharonov and Vaidman. They showed (Aharonov *et al.*, 1990) how superpositions of propagation processes (3.6) may arise when the system under investigation is coupled to a second system: a system influenced by an external system effectively evolves under a superposition of different connections. Furthermore (Aharonov and Vaidman, 1991), in terms of their language, they let a quantum system 1 be pre- and postselected in a superposition of product states with a second system 2; as far as system 1 is concerned, the pre- and postselection effectively can be described by a superposition of tensor products of a system-1-ket at the initial and of a system-1-bra at the final time of the experiment. This is an example of a superposition of coprocesses of the product type (3.4).

4. DIACHRONIC QUANTUM ACTION PRINCIPLE

The diachronic dynamical principle we envision (in particular for a fundamental quantum theory of space-time and matter) is a principle for quantum processes that takes on the same form in different quantum frames. Such a form-invariant dynamical principle does not single out an absolute frame. The absence of the latter is a relativity without correspondence in classical physics, called transformation theory by Dirac and quantum relativity by Finkelstein (1995, §1.2.2, §4.3.1). A "quantum-tensor" is an object that transforms tensorially under a change of quantum frame. An example is a process tensor, which represents quantum processes. On the playground of quantum mechanics, process tensors are elements of the bundle $\Pi\mathbb{T}^{N+1}$. A quantum frame α then consists of an orthogonal basis for each ket fiber, the dual bra bases, and the induced product bases for $\Pi\mathbb{T}^{N+1}$ and the dual space $\Pi^{\mathrm{D}}\mathbb{T}^{N+1}$; see (3.8). A diachronic dynamical principle needs to determine the amplitudes (up to a normalization and phase factor) that a dynamical process has in a particular frame.

The probability amplitude of an experiment is the contraction of a process with a coprocess tensor; see equation (3.10). Let the experimenter be passive at the intermediate times so that the coprocess tensor carries identity operators at these times. Then, only the basis elements

$$\alpha(a(N), \mathrm{t}_N, \ldots, a(2), a(2), \mathrm{t}_1, a(1), \mathrm{t}_0) \in \Pi(\mathrm{t}_N, \ldots, \mathrm{t}_0) \qquad (4.1)$$

for which the $a(n + 1)$ are equal to the $a'(n)$, have nonvanishing trace and can contribute to the transition amplitude. Following Finkelstein (1995, §12.5.2), we call these processes unbroken, the other ones broken. Now consider a basis system β for the process system that has been constructed out of other orthogonal ket bases $\{|\mathrm{t}, b'\rangle\}$ and their dual bra bases. A superposition of unbroken α processes generally involves broken processes when expanded in the basis system β. Hence, the principle of quantum relativity forces us to include broken processes in the formulation of diachronic quantum mechanics (as we have done from the start in order to allow for arbitrary intermediate acts).

We quantize the classical action principle for paths by taking the dynamical process and a coprocess as the quantum analogs of the family of dynamical paths and a selection of a dynamical path, respectively. Accordingly the quantum action should be a linear operator on the process space,

$$\begin{aligned} W\colon\quad \Pi\mathbb{T}^{N+1} &\to \Pi\mathbb{T}^{N+1} \\ \pi &\mapsto W\pi \end{aligned} \tag{4.2}$$

Let us call such a linear map a process operator. This concept of quantum action is suitable for a quantum theory in that it does not associate all quantum processes with well-defined action values. Similarly, input and outtake operations do not generally specify values of the configuration variables. Momentum and configuration variables are linear operators that generate configuration and momentum displacements of input and outtake vectors. This generator concept will reappear in the diachronic action principle: there the variation of the new quantum action will generate variations of the dynamical process tensor.

The quantum action will be expressed in terms of many-time configuration variables

$$q = \{q^1, \ldots, q^d\} \tag{4.3a}$$

The configuration frame χ consists of the configuration eigenkets and eigenbras and their tensor products

$$\begin{aligned} &\chi(q'(N), \mathrm{t}_N, \ldots, q(2), q'(1), \mathrm{t}_1, q(1), \mathrm{t}_0) \\ &\quad := \otimes \prod_{n=1}^{N} \left|q{:}q'(n), \mathrm{t}_n\rangle\langle q{:}q(n), \mathrm{t}_{n-1}\right| \in \Pi(\mathrm{t}_N, \ldots, \mathrm{t}_0) \end{aligned} \tag{4.3b}$$

$$\begin{aligned} &\chi^{\mathrm{D}}(q'(N), \mathrm{t}_N, \ldots, q(2), q'(1), \mathrm{t}_1, q(1), \mathrm{t}_0) \\ &\quad := \prod_{n=1}^{N} \langle q{:}q'(n), \mathrm{t}_n\right| \otimes \left|q{:}q(n), \mathrm{t}_{n-1}\rangle \in \Pi^{\mathrm{D}}(\mathrm{t}_N, \ldots, \mathrm{t}_0) \end{aligned} \tag{4.3c}$$

A process π is a superposition over basis processes, equation (4.3b), with different sequences of configuration values $q(n)$ and $q'(n)$, but with equal times t_n. In order to achieve a more symmetrical representation of the configuration degrees of freedom and of time, we combine processes with different sequences of initial, intermediate, and final times z_n into a section π of the process space,

$$\begin{aligned} \pi\colon\quad \mathbb{T}^{N+1} = \left\{(\mathrm{z}_N, \ldots, \mathrm{z}_0) \,\middle|\, t(\mathrm{z}_n) - t(\mathrm{z}_{n-1}) < 2\frac{T'}{N}\right\} &\to \Pi\mathbb{T}^{N+1} \\ (\mathrm{z}_N, \ldots, \mathrm{z}_0) &\mapsto \pi(\mathrm{z}_N, \ldots, \mathrm{z}_0) \end{aligned} \tag{4.4}$$

A coprocess σ over times $(\mathfrak{t}_N, \ldots, \mathfrak{t}_0)$ selects the process of the section at the same time sequence $(\mathfrak{t}_N, \ldots, \mathfrak{t}_0)$,

$$\sigma \in \Pi^{\mathrm{D}}(\mathfrak{t}_N, \ldots, \mathfrak{t}_0), \qquad \sigma\pi := \sigma\pi(\mathfrak{t}_N, \ldots, \mathfrak{t}_0) \tag{4.5}$$

A process operator acts on a process section by acting on the individual processes of the section.

Say we set $q'(n)$ equal to $q(n+1)$ in a configuration coprocess (4.3c). The intermediate acts at the intermediate times $(\mathfrak{t}_1, \ldots, \mathfrak{t}_{N-1})$ then are filter operations represented by the projectors $|q{:}q(n), \mathfrak{t}_n\rangle\langle q{:}q(n), \mathfrak{t}_n|$. Feynman called such an unbroken coprocess "an ideal measurement . . . performed to determine whether a particle has a path lying in a region of space-time." [Finkelstein (1995, §12.5.2) pointed out that "a determination of the quantum path is complementary to a determination of the Hamiltonian of the quantum, which includes its mass and other couplings." We are aware of this problem of the quantum mechanical concept of position preparations or measurements. In the present paper quantum mechanics serves as a playground to develop a new notion of quantum action.] For the corresponding probability amplitude Feynman proposes the sum over paths in that region weighted by the exponential of i times the action.

The principle of quantum relativity forces us to include broken coprocesses into the coprocess space [compare with equation (4.1)ff], as we have done from the start. Feynman's principle, formulated in terms of unbroken paths only, next is used to express the dynamical process tensor, which includes broken configuration processes. From that expression we shall obtain the quantum action and the differential diachronic action principle.

For an individual time step propagator Feynman's result yields,

$$U(\mathfrak{t}_n, \mathfrak{t}_{n-1}) \doteq \int |q{:}q'(n), \mathfrak{t}_n\rangle\, dq'(n) \times e^{(i/\hbar)W[\chi(q'(n),\mathfrak{t}_n,q(n),\mathfrak{t}_{n-1})]}\, dq(n)\langle q{:}q(n), \mathfrak{t}_{n-1}| \tag{4.6a}$$

where we have absorbed the normalization factor into the action. The tensor of the dynamical process over N time steps is the tensor product of the propagators over the various time steps; see equation (3.6). Hence

$$U(\mathfrak{t}_N, \ldots, \mathfrak{t}_0) \doteq \int \prod_{n=1}^{N} dq'(n)\, dq(n) \times e^{(i/\hbar)W[\chi(q'(N),\mathfrak{t}_N,\ldots,q(1),q'(1),\mathfrak{t}_1,q(0),\mathfrak{t}_0)]} \times \chi(q'(N), \mathfrak{t}_N, \ldots, q(2), q'(1), \mathfrak{t}_1, q(1), \mathfrak{t}_0) \tag{4.6b}$$

Here the action is defined for unbroken and broken paths as the sum of single step actions of equation (4.6a).

We restrict ourselves to the fairly general Lagrangian

$$L(t, q, v) = \frac{1}{2} vm(t)v - V(t, q) - \frac{A(t, q)v + vA(t, q)}{2} \tag{4.7a}$$

with the matrix m being real, symmetric, and invertible,

$$m^* = m = m^t, \qquad \det m \neq 0 \tag{4.7b}$$

The action of equations (4.6) then is (Lee, 1981)

$$\begin{aligned} &W[\chi(q'(N), t{:}t(N), \ldots, q(2), q'(1), t{:}t(1), q(1), t{:}t(0))] \\ &:= \sum_{n=1}^{N} \left\{ -\frac{i\hbar}{2} \ln \frac{\det m(t(n-1))}{(2\pi i\hbar[t(n) - t(n-1)])^d} \right. \\ &\quad + \frac{[q'(n) - q(n)]m(t(n-1))[q'(n) - q(n)]}{2[t(n) - t(n-1)]} \\ &\quad - \frac{A(t(n-1), q'(n))[q'(n) - q(n)] + [q'(n) - q(n)]A(t(n-1), q(n))}{2} \\ &\quad \left. - [t(n) - t(n-1)]V(t(n-1), q(n)) \right\} \end{aligned} \tag{4.8}$$

The quantum action W is defined to be a process operator that is diagonal in the configuration frame χ and possesses the actions of equation (4.8) as eigenvalues. Then W is a section of the bundle of operators that act within the process fibers,

$$\begin{aligned} &W(\mathrm{t}_N, \ldots, \mathrm{t}_0) \\ &:= \sum_{n=1}^{N} \left\{ -\frac{i\hbar}{2} \ln \frac{\det m(\mathrm{t}(t_{n-1}))}{(2\pi i\hbar[t(\mathrm{t}_n) - t(\mathrm{t}_{n-1})])^d} \right. \\ &\quad + \frac{[q(\mathrm{t}_n) - \overleftarrow{q}(\mathrm{t}_{n-1})]m(t(\mathrm{t}_{n-1}))[q(\mathrm{t}_n) - \overleftarrow{q}(\mathrm{t}_{n-1})]}{2[t(\mathrm{t}_n) - t(\mathrm{t}_{n-1})]} \\ &\quad - \left(\frac{A(t(\mathrm{t}_{n-1}), q(\mathrm{t}_n))[q(\mathrm{t}_n) - \otimes \overleftarrow{q}(\mathrm{t}_{n-1})]}{2} \right. \\ &\quad \left. + \frac{[q(\mathrm{t}_n) \otimes - \overleftarrow{q}(\mathrm{t}_{n-1})]A(t(\mathrm{t}_{n-1}), \overleftarrow{q}(\mathrm{t}_{n-1}))}{2} \right) \\ &\quad \left. - [t(\mathrm{t}_n) - t(\mathrm{t}_{n-1})]V(t(\mathrm{t}_{n-1}), \overleftarrow{q}(\mathrm{t}_{n-1})) \right\} \end{aligned} \tag{4.9}$$

Here we used two ways to act on $\Pi(\mathrm{t}_N, \ldots, \mathrm{t}_0)$ with an ordinary linear operator $a(\mathrm{t}_n) \in I(\mathrm{t}_n) \otimes I^{\mathrm{D}}(\mathrm{t}_n)$:

$$a(\mathrm{t}_n) \sum \cdots \otimes |\mathrm{t}_n, \psi\rangle\langle \mathrm{t}_{n-1}, \varphi| \otimes \cdots := \sum \cdots \otimes a(\mathrm{t}) |\mathrm{t}_n, \psi\rangle\langle \mathrm{t}_{n-1}, \varphi| \otimes \cdots \tag{4.10a}$$

$$\overleftarrow{a}(\mathrm{t}_n) \sum \cdots \otimes |\mathrm{t}_{n+1}, \alpha\rangle\langle \mathrm{t}_n, \beta| \otimes \cdots := \sum \cdots \otimes |\mathrm{t}_{n+1}, \alpha\rangle\langle \mathrm{t}_n, \beta| a(\mathrm{t}_n) \otimes \cdots \tag{4.10b}$$

Formula (4.6b) of a dynamical process tensor is easily expressed in terms of the quantum action,

$$\begin{aligned} U(\mathrm{t}_N, \ldots, \mathrm{t}_0) &\doteq e^{(i/\hbar)W} \int \prod_{n=1}^{N} dq'(n)\, dq(n) \\ &\quad \times \chi(q'(N), \mathrm{t}_N, \ldots, q(2), q'(1), \mathrm{t}_1, q(1), \mathrm{t}_0) \end{aligned} \tag{4.11}$$

From this we now derive a differential action principle for the process section U built out of the dynamical processes $U(\mathrm{t}_N, \ldots, \mathrm{t}_0)$. Under time and configuration displacements $\delta t(n)$, $\delta q(n)$, and $\delta q'(n)$, $n = 0, \ldots, N$, a process

$$\begin{aligned} \pi &= \sum c \cdots \otimes |q{:}q'(n), t{:}t(n)\rangle\langle q{:}q(n), t{:}t(n-1)| \otimes \cdots \\ &\in \Pi(t{:}t(N), \ldots, t{:}t(0)) \end{aligned} \tag{4.12a}$$

goes over into

$$\begin{aligned} \bar{\pi} &= \sum c \cdots \otimes |q{:}q'(n) + \delta q'(n), \bar{\mathrm{t}}_n\rangle\langle q{:}q(n) + \delta q(n), \bar{\mathrm{t}}_{n-1}| \otimes \cdots \\ &\in \Pi(\bar{\mathrm{t}}_N, \ldots, \bar{\mathrm{t}}_0) \end{aligned} \tag{4.12b}$$

The displaced times $\bar{\mathrm{t}}_n$ are defined to have time coordinate values $t + \delta t(n)$,

$$\bar{\mathrm{t}}_n = t{:}t(n) + \delta t(n) \tag{4.13}$$

We vary the action so that

$$\overline{W}\bar{\pi} = \overline{W\pi} \tag{4.14}$$

In terms of the displaced time and configuration variables the varied action takes on the same form as the unvaried one does in equation (4.9),

$$
\begin{aligned}
&\overline{W}(\bar{\mathrm{t}}_N, \ldots, \bar{\mathrm{t}}_0) \\
&:= \sum_{n=1}^{N} \left\{ -\frac{i\hbar}{2} \ln \frac{\det m(\bar{t}(\bar{\mathrm{t}}_{n-1}))}{(2\pi i\hbar[\bar{t}(\bar{\mathrm{t}}_n) - \bar{t}(\bar{\mathrm{t}}_{n-1})])^d} \right. \\
&\quad + \frac{[\bar{q}(\bar{\mathrm{t}}_n) - \overleftarrow{\bar{q}}(\bar{\mathrm{t}}_{n-1})] m(\bar{t}(\bar{\mathrm{t}}_{n-1}))[\bar{q}(\bar{\mathrm{t}}_n) - \overleftarrow{\bar{q}}(\bar{\mathrm{t}}_{n-1})]}{2[\bar{t}(\bar{\mathrm{t}}_n) - \bar{t}(\bar{\mathrm{t}}_{n-1})]} \\
&\quad - \left(\frac{A(\bar{t}(\bar{\mathrm{t}}_n), \bar{q}(\mathrm{t}_n))[\bar{q}(\bar{\mathrm{t}}_n) - \otimes\, \overleftarrow{\bar{q}}(\bar{\mathrm{t}}_{n-1})]}{2} \right. \\
&\quad + \left. \frac{[\bar{q}(\bar{\mathrm{t}}_n) \otimes - \overleftarrow{\bar{q}}(\bar{\mathrm{t}}_{n-1})] A(\bar{t}(\bar{\mathrm{t}}_{n-1}), \overleftarrow{\bar{q}}(\bar{\mathrm{t}}_{n-1}))}{2} \right) \\
&\quad \left. - [\bar{t}(\bar{\mathrm{t}}_n) - \bar{t}(\bar{\mathrm{t}}_{n-1})] V(\bar{t}(\bar{\mathrm{t}}_{n-1}), \overleftarrow{\bar{q}}(\bar{\mathrm{t}}_{n-1})) \right\}
\end{aligned} \tag{4.15a}
$$

$$
\begin{aligned}
&\bar{t}(\bar{\mathrm{t}}_n) = t(\mathrm{t}_n) - \delta t(n), \\
&\bar{q}(\bar{\mathrm{t}}_n) = q(\mathrm{t}_n) - \delta q'(n), \qquad \overleftarrow{\bar{q}}(\bar{\mathrm{t}}_{n-1}) = \overleftarrow{q}(\mathrm{t}_{n-1}) - \delta q(n)
\end{aligned} \tag{4.15b}
$$

(Note that we have varied q and $\overleftarrow{q}$ differently.) Equation (4.6b) transforms into

$$
\begin{aligned}
\overline{U}(\bar{\mathrm{t}}_N, \ldots, \bar{\mathrm{t}}_0) &\doteq e^{(i/\hbar)\overline{W}} \int \prod_{n=1}^{N} dq'(n)\, dq(n) \\
&\quad \times \chi(q'(N), \bar{\mathrm{t}}_N, \ldots, q(2), q'(1), \mathrm{t}_1, q(1), \bar{\mathrm{t}}_0)
\end{aligned} \tag{4.16}
$$

(The variations of the eigenvalues could be dropped because the integral extends over all eigenvalues.) The variation of the dynamical process section,

$$\delta U := \overline{U} - U \tag{4.17}$$

hence is generated by the variation of the action,

$$\delta U \doteq \frac{i}{\hbar}\, \delta W U \tag{4.18}$$

where

$$\delta W := \overline{W} - W \tag{4.19}$$

This equation not only holds for the kinematical variations considered so far, but also relates dynamical variations of U to the corresponding form variations of W, expressed here in terms of variations of the functions m, A, and V that appear in the Lagrangian L.

The section U represents the dynamical processes throughout the durations of the experiments. δW generates changes of a process at all times of the process. U, W, and their variations transform tensorially under changes of frame. Hence, equation (4.18) is a diachronic quantum-tensor equation. However, the quantum mechanical action is not quantum-relativistically invariant, in that it takes on a special form in the configuration frame χ. We brought a nonquantum-relativistic law into a covariant form in order to obtain a candidate for the form of a quantum-relativistic law of fundamental processes π. These processes π we suppose to underly the particle processes on a fixed background space-time. Then W is an operator on the fundamental process space that takes on the same form in different frames. The addition of different path amplitudes in quantum mechanics and of different virtual processes in field theory may be reminiscent of the linear structure of a fundamental process space. The limit to a field theory on a classical space-time structure then should be obtained via coherent states, some of whose parameters should be identifiable with space-time coordinates of the vertices of virtual processes.

Equation (4.18) determines the amplitudes $A(q'(N), \mathfrak{t}_N, \ldots, q(2), q'(1), \mathfrak{t}_1, q(1), \mathfrak{t}_0)$ of the dynamical processes in the configuration frame up to a common factor. In this frame equation (4.18) takes on the form

$$\begin{aligned} &A(\ldots, q(1) - \delta q(1), t{:}t(0) - \delta t(N)) - A(\ldots, q(1), t{:}t(0)) \\ &\quad \doteq \frac{i}{\hbar}(W[\chi(\ldots, q(1) - \delta q(1), t{:}t(0) - \delta t(N))] \\ &\quad\quad - W[\chi(\ldots, q(1), t{:}t(0))])A(\ldots, q(1), t{:}t(0)) \end{aligned} \tag{4.20}$$

This is easily integrated into

$$A(\ldots, q(1), t{:}t(0)) \doteq A_0 \exp\left\{\frac{i}{\hbar} W[\chi(\ldots, q(0), t{:}t(0))]\right\} \tag{4.21}$$

which agrees with equation (4.6b).

5. PICTURE-INDEPENDENT SCHWINGER ACTION PRINCIPLE

It is instructive to see how Schwinger's action principle is related to our diachronic one. The trace of the diachronic action principle, equation (4.18), is

$$\delta U(\mathfrak{t}_N, \mathfrak{t}_0) \doteq \frac{i}{\hbar} \operatorname{Tr}(\delta W U)(\mathfrak{t}_N, \mathfrak{t}_0) \tag{5.1}$$

We shall let the final variations $\delta q'(n)$ of the nth time step be equal to the initial variation $\delta q(n+1)$ of the $(n+1)$th step. The contracted variation $\delta U(\mathrm{t}_N, \mathrm{t}_0)$ then only depends on the endpoint variations $\delta t(0)$, $\delta q(1)$, $\delta t(N)$, and $\delta q'(N)$. Hence the right side of equation (5.1) also does not depend on δt and δq at intermediate times. This condition will yield equations of motion.

Let us expand the kinematical variation of the action $W(n)$ over a single time step n to the first order in $\delta t(n-1)$, $\delta q(n)$, $\delta t(n)$, and $\delta q'(n)$,

$$
\begin{aligned}
\delta W(n) = \Bigg\{ & -\frac{i\hbar d}{2[t(\mathrm{t}_n) - t(\mathrm{t}_{n-1})]} \\
& + \frac{[q(\mathrm{t}_n) - \overleftarrow{q}(\mathrm{t}_{n-1})] m(t(\mathrm{t}_{n-1}))[q(\mathrm{t}_n) - \overleftarrow{q}(\mathrm{t}_{n-1})]}{2[t(\mathrm{t}_n) - t(\mathrm{t}_{n-1})]^2} \\
& + V(t(\mathrm{t}_{n-1}), \overleftarrow{q}(\mathrm{t}_{n-1}))\Bigg\} \\
& \times [\delta t(n) - \delta t(n-1)] \\
& + \Bigg\{ \frac{i\hbar}{2} \operatorname{Tr}[\dot{m}(t(\mathrm{t}_{n-1}))m^{-1}(t(\mathrm{t}_{n-1}))) \\
& - \frac{[q(\mathrm{t}_n) - \overleftarrow{q}(\mathrm{t}_{n-1})]\dot{m}(t(\mathrm{t}_{n-1}))[q(\mathrm{t}_n) - \overleftarrow{q}(\mathrm{t}_{n-1})]}{2[t(\mathrm{t}_n) - t(\mathrm{t}_{n-1})]} \\
& + \frac{\dot{A}(t(\mathrm{t}_{n-1}), q(\mathrm{t}_n))[q(\mathrm{t}_n) - \otimes \overleftarrow{q}(\mathrm{t}_{n-1})]}{2} \\
& + \frac{[q(\mathrm{t}_n) \otimes - \overleftarrow{q}(\mathrm{t}_{n-1})]\dot{A}(t(\mathrm{t}_{n-1}), \overleftarrow{q}(\mathrm{t}_{n-1}))}{2} \\
& + [t(\mathrm{t}_n) - t(\mathrm{t}_{n-1})]\dot{V}(t(\mathrm{t}_{n-1}), \overleftarrow{q}(\mathrm{t}_{n-1}))\Bigg\}\delta t(n-1) \\
& + \Bigg\{ -\frac{[q(\mathrm{t}_n) - \overleftarrow{q}(\mathrm{t}_{n-1})]}{[t(\mathrm{t}_n) - t(\mathrm{t}_{n-1})]} m(t(\mathrm{t}_{n-1})) \\
& + \frac{A(t(\mathrm{t}_{n-1}), q(\mathrm{t}_n)) + A(t(\mathrm{t}_{n-1}), \overleftarrow{q}(\mathrm{t}_{n-1}))}{2}\Bigg\}[\delta q'(n) - \delta q(n)]
\end{aligned}
$$

$$+ \delta q'(n) \frac{\partial_q A(t(\mathrm{t}_{n-1}), q(\mathrm{t}_n))[q(\mathrm{t}_n) - \otimes \overleftarrow{q}(\mathrm{t}_{n-1})]}{2}$$

$$+ \left\{ \frac{[q(\mathrm{t}_n) \otimes - \overleftarrow{q}(\mathrm{t}_{n-1})]\partial_q A(t(\mathrm{t}_{n-1}), \overleftarrow{q}(\mathrm{t}_{n-1}))}{2} \right.$$

$$\left. + [t(\mathrm{t}_n) - t(\mathrm{t}_{n-1})]\partial_q V(t(\mathrm{t}_{n-1}), \overleftarrow{q}(\mathrm{t}_{n-1})) \right\} \delta q(n) \tag{5.2}$$

We sum over the different time steps, act on the dynamical process section U, take the trace, and set $\delta q'(n) = \delta q(n + 1)$ for $n = 1, \ldots, N - 1$. The result $\mathrm{Tr}(\delta WU)$ must not depend on the intermediate variations $\delta t(n)$ and $\delta q'(n) \equiv \delta q(n + 1)$ with $n = 1, \ldots, N - 1$. For large N the consequent dynamical equations take on the form

$$\left.\frac{d}{dt}\right|_U H \doteq -\frac{1}{2}\,\dot{q}_{t,U}\dot{m}\dot{q}_{t,U} + \frac{i\hbar\,\mathrm{Tr}[\dot{m}m^{-1}] - [q^j, \dot{q}^k_{t,U}\dot{m}_{kj}]}{2[t(\mathrm{t}_n) - t(\mathrm{t}_{n-1})]} + \dot{V} + \frac{\dot{A}v + v\dot{A}}{2}$$

$$H := \frac{1}{2}\,\dot{q}_{t,U}m\dot{q}_{t,U} + V - \frac{i\hbar d - [q^j, \dot{q}^k_{t,U}m_{kj}]}{2[t(\mathrm{t}_n) - t(\mathrm{t}_{n-1})]} \tag{5.3a}$$

$$\left.\frac{d}{dt}\right|_U p := -\partial_q V - \frac{\partial_q A v + v\partial_q A}{2}$$

$$p := \dot{q}_{t,U}m - A \tag{5.3b}$$

[When q and $\overleftarrow{q}$ act as process operators—see (4.10)—quadratic expressions of them are effectively time ordered, whereas a quadratic expression of the ordinary operators $\dot{q}_{t,U}$ is not time ordered when expressed in terms of $q(\mathrm{t}_{n-1})$ and $q(\mathrm{t}_n)$. The commutators of q with $\dot{q}_{t,U}$ originated from this difference.] The total time derivatives that appear here are covariant derivatives [see equation (2.9)] with respect to the dynamical connection, which for large N is well represented by the dynamical process tensor U. Up to boundary terms of order $\delta t/N$ and $\delta q'/N$ we find

$$\begin{aligned} &\mathrm{Tr}(\delta WU)(\mathrm{t}_N, \mathrm{t}_0) \\ &\quad \doteq [H(\mathrm{t}_N)\delta t(N) - p(\mathrm{t}_N)\delta q'(N)]U(\mathrm{t}_N, \mathrm{t}_0) \\ &\qquad - U(\mathrm{t}_N, \mathrm{t}_0)[H(\mathrm{t}_0)\delta t(0) - p(\mathrm{t}_0)\delta q(0)]] \end{aligned} \tag{5.4}$$

According to (4.12), the configuration matrix elements of the variation of the time evolution operator are

$$\begin{aligned}
&\langle q{:}q(N),\, t{:}t(N)|\,\delta U(t{:}t(N),\, t{:}t(0))\,|q{:}q(0),\, t{:}t(0)\rangle \\
&= \langle q{:}q(N) - \delta q'(N),\, t{:}t(N) - \delta t(N)| \\
&\quad \times U(t{:}t(N) - \delta t(N),\, t{:}t(0) - \delta t(0)) \\
&\quad |q{:}q(0) - \delta q(0),\, t{:}t(0) - \delta t(0)\rangle \\
&\quad - \langle q{:}q(N),\, t{:}t(N)|\,U(t{:}t(N),\, t{:}t(0))\,|q{:}q(0),\, t{:}t(0)\rangle \\
&=: \delta\langle q{:}q(N),\, t{:}t(N)|\,U(t{:}t(N),\, t{:}t(0))\,|q{:}q(0),\, t{:}t(0)\rangle
\end{aligned} \tag{5.5}$$

The matrix elements of equation (5.1) hence are

$$\begin{aligned}
&\delta\langle q{:}q(N),\, \mathrm{t}_N|\,U(\mathrm{t}_N,\, \mathrm{t}_0)\,|q{:}q(0),\, \mathrm{t}_0\rangle \\
&\doteq \frac{i}{\hbar}\,\langle q{:}q(N),\, \mathrm{t}_N|\,\mathrm{Tr}(\delta WU)(\mathrm{t}_N,\, \mathrm{t}_0)\,|q{:}q(0),\, \mathrm{t}_0\rangle
\end{aligned} \tag{5.6}$$

Together with equation (5.4), this implies the canonical commutation relations

$$[p_j,\, q^k] \doteq -i\hbar\delta_j^k, \qquad j, k = 1, \ldots, d \tag{5.7}$$

and the Heisenberg equations,

$$\dot{q}_{t,U} \doteq \frac{i}{\hbar}\,[H,\, q] \tag{5.8}$$

The canonical commutation relations imply the following commutation relation between the velocities and the configuration variables:

$$[\dot{q}^k_{t,U},\, q^j] \doteq -i\hbar m^{-1kj}(t) \tag{5.9}$$

Using this, we simplify (5.3a) into

$$\begin{aligned}
\left.\frac{d}{dt}\right|_U H &\doteq -\frac{1}{2}\,\dot{q}_{t,U}\dot{m}\dot{q}_{t,U} + \dot{V} + \frac{\dot{A}v + v\dot{A}}{2} \\
H &= \frac{1}{2}\,\dot{q}_{t,U}m\dot{q}_{t,U} + V
\end{aligned} \tag{5.10}$$

We now can see [compare with DeWitt's derivation of the Schwinger action principle at the end of DeWitt (1957)] that $\mathrm{Tr}(\delta WU)(\mathrm{t}_N,\, \mathrm{t}_0)$ is equal to the variation of the picture-independent form of the Schwinger action (Schwinger, 1960, 1970; Mantke, 1992),[3,4]

[3] Schwinger includes time variations in the class of dynamical changes, whereas we call variations of time kinematical, as we do with variations of configuration.

[4] Note that Mantke failed to notice that the picture-independent Schwinger action does not act linearly on the dynamical connection because of the square velocity term of the Lagrangian.

$$\mathrm{Tr}(\delta WU)(\mathrm{t}_N, \mathrm{t}_0) \doteq \delta W_S[U](\mathrm{t}_N, \mathrm{t}_0) \tag{5.11a}$$

$$W_S[U](\mathrm{tyN}, \mathrm{t}_0) := \int_{t(\mathrm{t}_0)}^{t(\mathrm{t}_N)} dt' \; U(\mathrm{t}_N, t{:}t')L(t, q, \dot{q}_{t,U})U(t{:}t', \mathrm{t}_0) \tag{5.11b}$$

W_S is a connection functional with values in $I(\mathrm{t}_N) \otimes I^{\mathrm{D}}(\mathrm{t}_0)$. As for W in equations (4.15), W_S is to be varied by replacing t (including dt) and q with

$$\bar{t} := t - \delta t(t) \qquad \text{and} \qquad \bar{q} := q - \delta q(t) \tag{5.11c}$$

The endpoint variations in equation (5.5) we now express as

$$\begin{aligned} \delta t(0) &= \delta t(t(\mathrm{t}_n)), & \delta q(0) &= \delta q(t(\mathrm{t}_0)) \\ \delta t(N) &= \delta t(t(\mathrm{t}_N)), & \delta q'(N) &= \delta q(t(\mathrm{t}_N)) \end{aligned} \tag{5.11d}$$

Combining equations (5.6) and (5.11a), we arrive at the picture-independent form of the Schwinger action principle

$$\begin{aligned} &\delta\langle q{:}q(N), \mathrm{t}_N | U(\mathrm{t}_N, \mathrm{t}_0) | q{:}q(0), \mathrm{t}_0\rangle \\ &\doteq \frac{i}{\hbar} \langle q{:}q(N), \mathrm{t}_N | \delta W_S[U](\mathrm{t}_N, \mathrm{t}_0) | q{:}q(0), \mathrm{t}_0\rangle \end{aligned} \tag{5.12}$$

There is an alternative form to express the variations involved in equation (5.12): Let us express W_S, equation (5.11b), in terms of the variables $t^* = t + \delta t(t)$ (including dt^*) and $q^* = q + \delta q(t)$. The resulting form variation of W_S is equal to δW_S. Similarly, the variation of the transition amplitude results from replacing eigenvectors $|q{:}q', t{:}t'\rangle$ of t and q by eigenvectors $|q^*{:}q', t^*{:}t'\rangle$ of t^* and q^* with equal eigenvalues.

We have shown how the diachronic action principle implies the picture-independent form of the Schwinger action principle for kinematical variations. A similar derivation holds for dynamical variations. (The dynamical variation of the transition amplitude is the matrix element of the dynamical variation of the evolution operator. The dynamical variation of the action is the form variation which represents the change of dynamics.)

6. CONCLUSIONS

Feynman's integral action principle is diachronic because the paths in terms of which it is formulated are concepts that extend throughout the time interval of the considered experiment; the principle is not quantum relativistic, because it considers unbroken paths only. After a change of quantum frame an unbroken path, or process in general, is expressed as a superposition of broken processes. Schwinger employed the Heisenberg picture to formulate his differential action principle. Thus his principle is not diachronic, in that it does not involve an explicit representation of the dynamical process during an experiment. The Schwinger action principle transforms covariantly when we pass from the configuration frame to another frame. (We keep the matrix of that transformation constant as we vary the configuration eigenvectors. Their variations then transform into the appropriate variation of the new basis vectors.)

The diachronic quantum action principle combines the diachronic feature of Feynman's principle with the quantum covariance of Schwinger's. Only the endpoint variations of the configuration and time coordinates affect the transition amplitude, so that the Schwinger action principle and its picture-independent formulation assign no experimental meaning to the intermediate variations of the variables. The diachronic action principle remedies this: the variation of the quantum action generates the displacements of the dynamical process at all times of the process.

The form of the diachronic action principle is a candidate form for the dynamical law of a theory without a classical time parameter. Our action acts on quantum processes as a linear operator and the variation of the action generates variations of dynamical processes. This approach appears to be applicable to a theory of fundamental processes, such as Finkelstein's (1972a,b, 1995) chronons, which we suppose to underly our concept and experience of particle processes.

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